# A long $\mathbb{C}^n$ without holomorphic functions

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(This is joint work with Franc Forstnerič)

# Introduction

- Behnke-Thullen (1933) raised the following question:
  *Is an increasing union of Stein domains in $\mathbb{C}^n$ always Stein?* **Yes!** (1939)

- Union problem: *Is an increasing union of Stein manifolds always Stein?* **No!**
  Counterexample was given in any dimension $n \geq 3$ by J. E. Fornæss in (1976); he found an increasing union of balls that is not holomorphically convex, hence not Stein.

  The key ingredient in his proof is a construction of a biholomorphic map $\Phi\colon \Omega \to \Phi(\Omega) \subset \mathbb{C}^3$ on a bounded neighborhood $\Omega \subset \mathbb{C}^3$ of any compact set $K \subset \mathbb{C}^3$ with nonempty interior such that the polynomial hull of $\Phi(K)$ is not contained in $\Phi(\Omega)$. (A phenomenon of this type was first described by Wermer in 1959)

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# Definitions

- Suppose we have a system $\{(X_k, f_k)\}_k$ of $n$-dimensional complex manifolds $X_k$ and holomorphic embeddings $f_k : X_k \to X_{k+1}$. The direct limit of such system is a $n$-dimensional complex manifold $X$.
- We say that $X$ is a **short** $\mathbb{C}^n$ if every $X_k$ is equal to $\mathbb{B}^n$, i.e. a complex manifold which is exhausted by holomorphic embeddings of $n$-dimensional Euclidean ball.
- Short $\mathbb{C}^n$'s are quite natural objects and they were studied by Fornæss, Sibony and others.
- We say that $X$ is a **long** $\mathbb{C}^n$ if every $X_k$ is equal to $\mathbb{C}^n$, i.e. a complex manifold which is exhausted by holomorphic embeddings of $n$-dimensional Euclidean space.
- Observe that every long $\mathbb{C}^n$ is also a short $\mathbb{C}^n$ and an Oka manifold, but the not vice versa.

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# Non-Stein long $\mathbb{C}^2$

The first counterexample to the union problem in dimension $n = 2$ was the result of E. Wold (2010) on the existence of a non-Stein long $\mathbb{C}^2$.

The main idea behind the construction is similar to one in J.E. paper, but here the key ingredient is the existence of a holomorphic embedding $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\Phi(\mathbb{C}^2)$ is not Runge in $\mathbb{C}^2$.This was also done by Wold (2007) using:

- Stolzenberg contruction of two disjoint smooth, embedded, totally real discs, whose union is holomorphically convex in $\mathbb{C}^* \times \mathbb{C}$ but it is not polynomially convex, moreover $(0,0) \in \widehat{Y}$
- Andersén–Lempert theory

**Theorem (Wold process)**

*For every compact set with nonempty interior $K \subset \mathbb{C}^* \times \mathbb{C}$ there exist $\Phi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ satisfying $Y \subset \Phi(K)$, i.e. $\widehat{\Phi(K)} \cap \{0\} \times \mathbb{C} \neq \emptyset$.*

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## Stein long $\mathbb{C}^n$

**Theorem (Wold)**

*Suppose that $X_k = \mathbb{C}^n$ and $f_k(X_k)$ is Runge in $X_{k+1}$ for every $k > 0$, then $X$ is biholomorphic to $\mathbb{C}^n$*

**Open problem**

Does there exist a long $\mathbb{C}^n$ for any $n > 1$ which is a Stein manifold different from $\mathbb{C}^n$?

**Theorem**

*There exist a sequence of holomorphic embeddings $f_k : \mathbb{C}^n \to \mathbb{C}^n$ such that $f_k(\mathbb{C}^n)$ is not Runge in $\mathbb{C}^n$, but the induced long $\mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$.*

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## Some more problems

- Is there a domain in $\mathbb{C}^n$, which is a long $\mathbb{C}^n$ but is different from $\mathbb{C}^n$?
- Does there exist many non-biholomorphic non-Stein long $\mathbb{C}^n$'s?
- Does there exist a long $\mathbb{C}^n$ which admits no nonconstant holomorphic functions?
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# Long $\mathbb{C}^n$ which admits no nonconstant holomorphic functions

## Lemma

*Let $K$ be a compact set with nonempty interior in $\mathbb{C}^n$ for some $n > 1$. For every point $a \in \mathbb{C}^n$ there exists an injective holomorphic map $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that the polynomial hull of the set $\phi(K)$ contains the point $\phi(a)$. More generally, if $L \subset \mathbb{C}^n$ is a compact holomorphically contractible set disjoint from $K$ such that $K \cup L$ is polynomially convex, then there exists an injective holomorphic map $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that $\phi(L) \subset \widehat{\phi(K)}$ and $\widehat{\phi(K)} \setminus \phi(\mathbb{C}^n) \neq \emptyset$.*

## Theorem

*For every $n > 1$ there exist a long $\mathbb{C}^n$ which admits no non-constant holomorphic functions.*

**The idea of the proof:** Setting $X_i = \mathbb{C}^n$, $B \subset X_1$ closed unit ball and $\{a_k\}_k$ a dense subset of $X_1$ we construct $f_i : X_i \to X_{i+1}$ holomorphic embeddings such that

$$f_i \circ \ldots \circ f_1(a_i) \in \widehat{f_i(B_i)}$$

for all $i \geq 1$, where $B_i = f_{i-1} \circ \widehat{\ldots \circ} f_1(B)$.

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# The Stable Hull Property

**Definition (The Stable Hull Property)**

A compact set $B$ in a complex manifold $X$ has the *stable hull property*, SHP, if there exists an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$ by compact sets such that $B \subset K_1$, $K_j \subset \mathring{K}_{j+1}$ for every $j \in \mathbb{N}$, and the increasing sequence of hulls $\widehat{B}_{\mathscr{O}(K_j)}$ stabilizes, i.e., there is a $j_0 \in \mathbb{N}$ such that

$$\widehat{B}_{\mathscr{O}(K_j)} = \widehat{B}_{\mathscr{O}(K_{j_0})} \quad \text{for all } j \geq j_0. \tag{1}$$

- Obviously, SHP is a biholomorphically invariant property: if a compact set $B \subset X$ satisfies condition (1) with respect to some exhaustion $(K_j)_{j\in\mathbb{N}}$ of $X$, then for any biholomorphic map $F\colon X \to Y$ the set $F(B) \subset Y$ satisfies (1) with respect to the exhaustion $L_j = F(K_j)$ of $Y$.
- What is less obvious, but needed to make this condition useful, is its independence of the choice of the exhaustion.

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- What is less obvious, but needed to make this condition useful, is its independence of the choice of the exhaustion.

# Independence of the choice of the exhaustion

Set $C := \widehat{B}_{\mathscr{O}(K_{j_0})}$. Let $(L_l)_{l\in\mathbb{N}}$ be another exhaustion of $X$ by compact sets satisfying $L_l \subset \mathring{L}_{l+1}$ for all $l \in \mathbb{N}$. Pick an integer $l_0 \in \mathbb{N}$ such that $C \subset L_{l_0}$. Since both sequences $\mathring{K}_j$ and $\mathring{L}_l$ exhaust $X$, we can find sequences of integers $j_1 < j_2 < j_3 < \cdots$ and $l_1 < l_2 < l_3 < \cdots$ such that $j_0 \le j_1$, $l_0 \le l_1$, and

$$K_{j_0} \subset L_{l_1} \subset K_{j_1} \subset L_{l_2} \subset K_{j_2} \subset L_{l_3} \subset \cdots$$

From this and (1) we obtain

$$C = \widehat{B}_{\mathscr{O}(K_{j_0})} \subset \widehat{B}_{\mathscr{O}(L_{l_1})} \subset \widehat{B}_{\mathscr{O}(K_{j_1})} = C \subset \widehat{B}_{\mathscr{O}(L_{l_2})} \subset \widehat{B}_{\mathscr{O}(K_{j_2})} = C \subset \cdots .$$

It follows that $\widehat{B}_{\mathscr{O}(L_{l_j})} = C$ for all $j \in \mathbb{N}$. Since the sequence of hulls $\widehat{B}_{\mathscr{O}(L_l)}$ is increasing with $l$, we conclude that

$$\widehat{B}_{\mathscr{O}(L_l)} = C \quad \text{for all } l \ge l_1.$$

Hence, $B$ has the stable hull property with respect to the exhaustion $(L_l)_{l\in\mathbb{N}}$ of $X$.

# Stable core

## Definition

Let $X$ be a complex manifold.

(i) The *stable core* of $X$, denoted $SC(X)$, is the open set consisting of all points $x \in X$ which admit a compact neighborhood $K \subset X$ with the stable hull property.

(ii) A regular compact set $B \subset X$ is called the *strongly stable core* of $X$, denoted $SSC(X)$, if $B$ has the stable hull property, but no compact set $K \subset X$ with $\mathring{K} \setminus B \neq \emptyset$ has the stable hull property.

- Clearly, the stable core always exists and is a biholomorphic invariant, in the sense that any biholomorphic map $X \to Y$ maps $SC(X)$ onto $SC(Y)$. In particular, every holomorphic automorphism of $X$ maps the stable core $SC(X)$ onto itself.
- The strongly stable core $SSC(X)$ need not exist in general; if it does, then its interior equals the stable core $SC(X)$ and $SSC(X) = \overline{SC(X)}$.

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(ii) A regular compact set $B \subset X$ is called the *strongly stable core* of $X$, denoted $SSC(X)$, if $B$ has the stable hull property, but no compact set $K \subset X$ with $\mathring{K} \setminus B \neq \emptyset$ has the stable hull property.

- Clearly, the stable core always exists and is a biholomorphic invariant, in the sense that any biholomorphic map $X \to Y$ maps $SC(X)$ onto $SC(Y)$. In particular, every holomorphic automorphism of $X$ maps the stable core $SC(X)$ onto itself.
- The strongly stable core $SSC(X)$ need not exist in general; if it does, then its interior equals the stable core $SC(X)$ and $SSC(X) = \overline{SC(X)}$.

# Stable core

**Theorem**

*Let $n > 1$.*

(a) *For every regular compact polynomially convex set $B \subset \mathbb{C}^n$ (i.e., $B = \overline{\mathring{B}}$) there exists a long $\mathbb{C}^n$, $X(B)$ whose strongly stable core equals $B$: $SSC(X(B)) = B$.*

(b) *For every open set $U \subset \mathbb{C}^n$ there exists a long $\mathbb{C}^n$, $X$, which satisfies $SC(X) \subset U$ and $\overline{U} = \overline{SC(X)}$.*

**Corollary**

*Let $n > 1$. To every regular compact polynomially convex set $B \subset \mathbb{C}^n$ we can associate a complex manifold $X(B)$, which is a long $\mathbb{C}^n$ containing a biholomorphic copy of $B$, such that every biholomorphic map $\Phi\colon X(B) \to X(C)$ between two such manifolds takes $B$ onto $C$.*

Since $C = SSC(X(C))$ and $B$ has a SHP we have $\Phi(B) \subset C$. Since $B = SSC(X(B))$ and $C$ has a SHP we have $\Phi^{-1}(C) \subset B$, and hence $\Phi(B) = C$.

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# Proof (a)

At every step of the recursion we perform the Wold process simultaneously on finitely many pairwise disjoint compact sets $K_1, \dots, K_m$ in the complement of the given smoothly bounded, strongly pseudoconvex, polynomially convex compact set $B \subset \mathbb{C}^n$, chosen such that $\bigcup_{j=1}^m K_j \cup B$ is polynomially convex, thereby ensuring that polynomial hulls of their images $\phi(K_j)$ escape from the range of the injective holomorphic map $\phi \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ constructed in the recursive step.

At the same time, we ensure that $\phi$ is close to the identity map on a neighborhood of $B$, and hence the image $\phi(B)$ remains polynomially convex. In practice, the sets $K_j$ will be small pairwise disjoint closed balls in the complement of $B$ whose number will increase during the process.

# Proof (a)

We devise the process so that every point in a certain countable dense set $A = \{a_j\}_{j=1}^{\infty} \subset X \setminus B$ is the center of a decreasing sequence of balls whose $\mathscr{O}(X_k)$-hulls escape from each compact set in $X$; hence none of these balls has the stable hull property.

Since every compact polynomially convex set $B \subset \mathbb{C}^n$ is the intersection $B = \cap_{j=1}^{\infty} B_j$ of a decreasing sequence $B_1 \supset B_2 \supset \ldots \supset B$ of compact polynomially convex and smoothly bounded, strongly pseudoconvex domains, we can perform the construction such that at the k-th step the image of $B_k$ remains polynomially convex, but the Wold process starts taking place in $\mathbb{C}^n \backslash B_k$. Hence we can ensure that $B$ is the strongly stable core of the limit manifold $X$.

# Proof (b)

To prove part (b), we modify the recursion by introducing a new small ball $B' \subset U \setminus B$ at every stage. Thus, the set $B$ acquires additional connected components during the recursive process. The sequence of added balls $B_l$ is chosen such that their union is dense in the given open subset $U \subset \mathbb{C}^n$, while the sequence of sets $K_j$ on which the Wold process is performed densely fills the complement $X \setminus \overline{U}$. It follows that the stable core of the limit manifold $X = \bigcup_{k=1}^{\infty} X_k$ is contained in $U$ and is everywhere dense in $U$.

THANK YOU